Counting Paths in Quantum Logspace

Simon Apers, Roman Edenhofer

Algorithms and Complexity, IRIF

edenhofer@irif.fr

May 8, 2024

PCQT-IQC workshop on Algorithms and Complexity

Agenda

1. Space-bounded Computation
2. Graph Connectivity
3. References

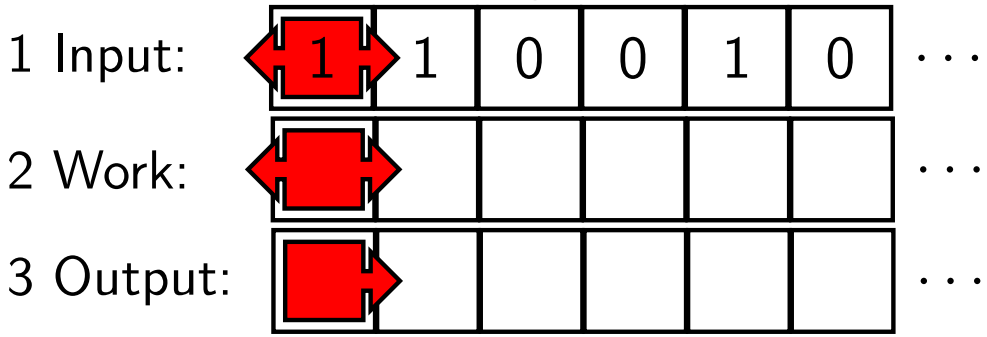
1 Space-bounded Computation

2 Graph Connectivity

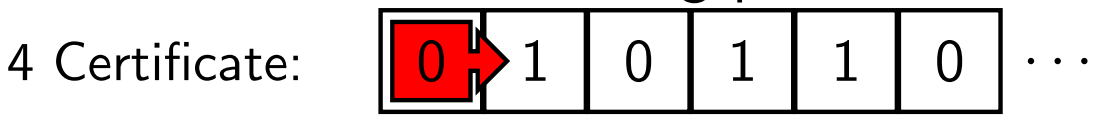
3 References

Complexity Classes

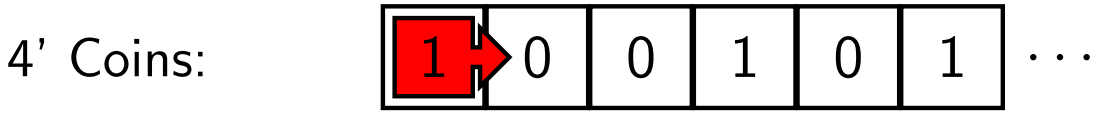
- **L** = Deterministic Logspace



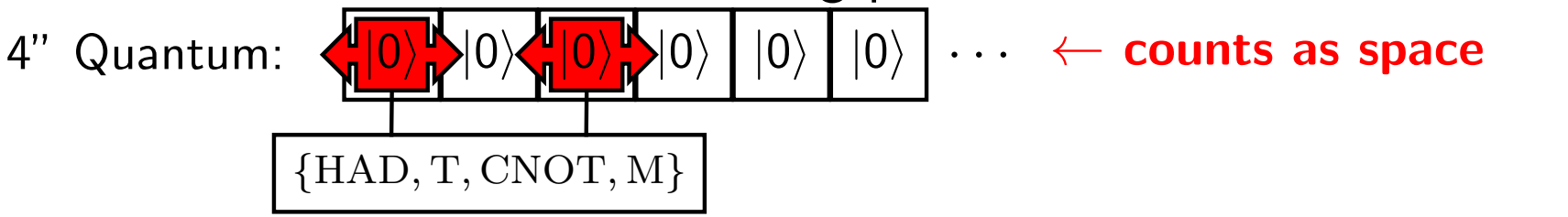
- **NL** = Non-deterministic Logspace



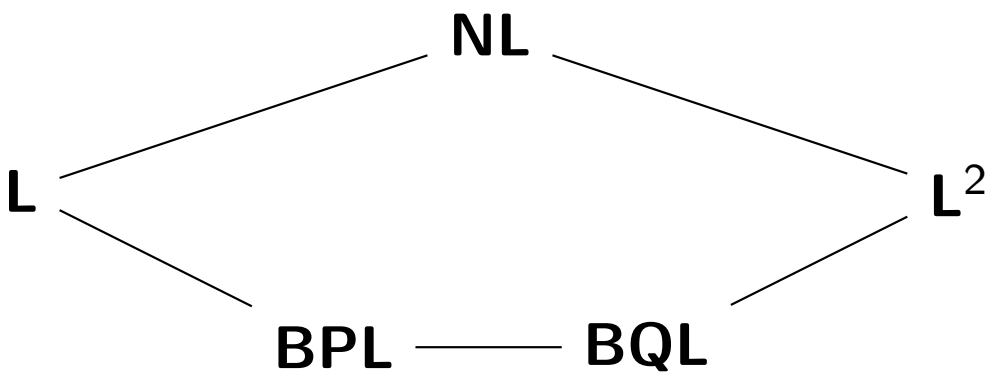
- **BPL** = Bounded-error Probabilistic Logspace



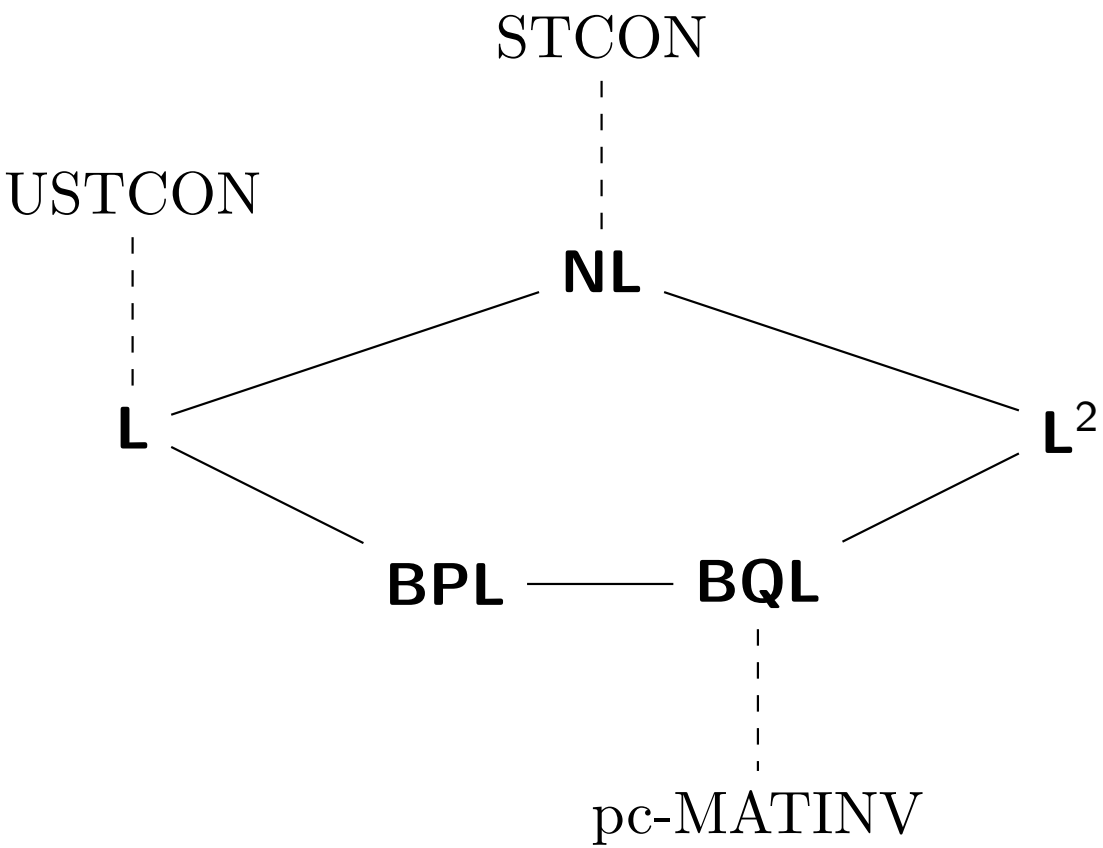
- **BQL** = Bounded-error Quantum Logspace



Complexity Classes



Complexity Classes

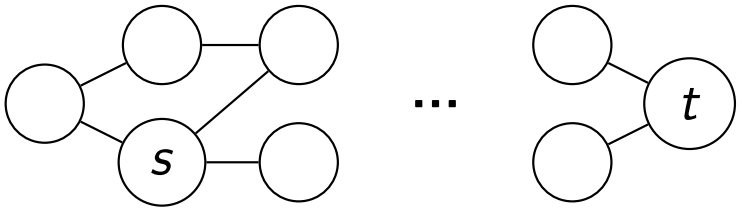


1 Space-bounded Computation

2 Graph Connectivity

3 References

USTCON



$$P(i,j) := \begin{cases} \frac{1}{d(j)} & \text{if } \{i,j\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \mu_t = P^t \mu_0$$

Facts

1. $\forall i\colon |\lambda_i(P)| \leq 1.$
2. G connected & non-bip $\Rightarrow \begin{cases} \exists!\text{ stationary } \pi : P\pi = \pi, \\ |\lambda_2| < 1. \end{cases}$

$$\Rightarrow P^t = 1 \cdot |\pi\rangle\langle\pi| + \underbrace{\sum_{j \geq 2} \lambda_j^t |v_j\rangle\langle v_j|}_{\to 0}.$$

$$\text{In fact } \begin{cases} \pi(i) = \frac{d(i)}{\sum_j d(j)}, \\ \delta := 1 - |\lambda_2| \geq 1/n^3. \end{cases}$$

USTCON

BPL-Procedure for USTCON

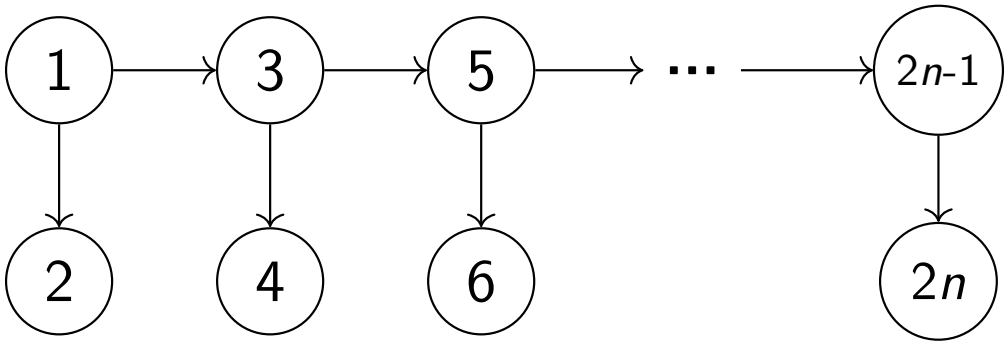
1. Do many Random Walks

BQL-Procedure for USTCON

1. Ta-Shma [1]: In **BQL** can do $\mathrm{QPE}(e^{iP})$ on random $|v\rangle_I$
   $\Rightarrow$ estimate $\dim(\ker(\mathrm{I}-\mathrm{P})) = \#\mathrm{CCs}$
2. Does dim change if we add edge $\{s,t\}$?

STCON

Problematic example



- In general $\delta := 1 - s_2$ exp-small
  $\Rightarrow$ Estimating $\dim(\ker(I - P))$ doesn't work anymore ☹

Theorem (Ta-Shma [1])

If A poly-conditioned, i.e. $\mathrm{poly}(n) \geq s_1(A) \geq \ldots \geq s_n(A) \geq \frac{1}{\mathrm{poly}(n)}$, then can approximate A^{-1} up to $\frac{1}{\mathrm{poly}(n)}$ accuracy in **BQL**.

Ta-Shma for STCON

Theorem

If $\#\text{paths}(i,j) \leq \text{poly}(n)\ \forall i,j$, then can decide STCON in **BQL**.

Proof.

Consider $\mathcal{L} := I - A$ for adjacency matrix A of a DAG.

$\Rightarrow\ \mathcal{L}^{-1} = I + A + ... + A^{n-1}$ because $A^n = 0$,

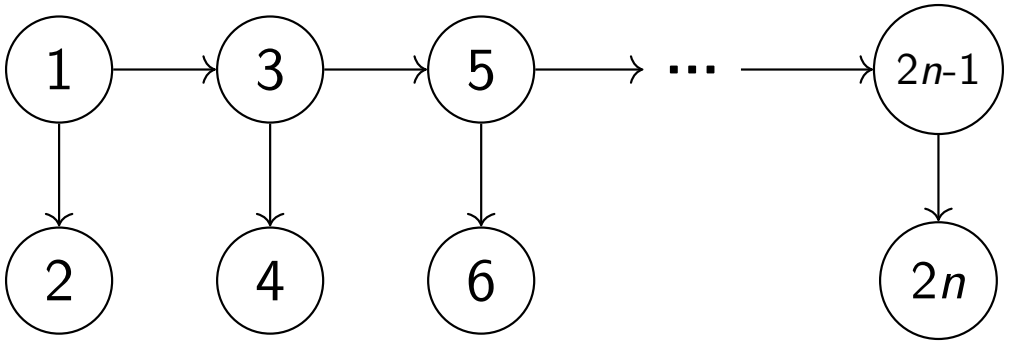
$\Rightarrow\ \mathcal{L}^{-1}(i,j) = \#\text{paths}(i,j)$.

Hence, $\mathcal{L}$ is poly-conditioned iff $\begin{cases} ||\mathcal{L}||_\infty \leq \text{poly}(n), \\ ||\mathcal{L}^{-1}||_\infty \leq \text{poly}(n), \end{cases}$

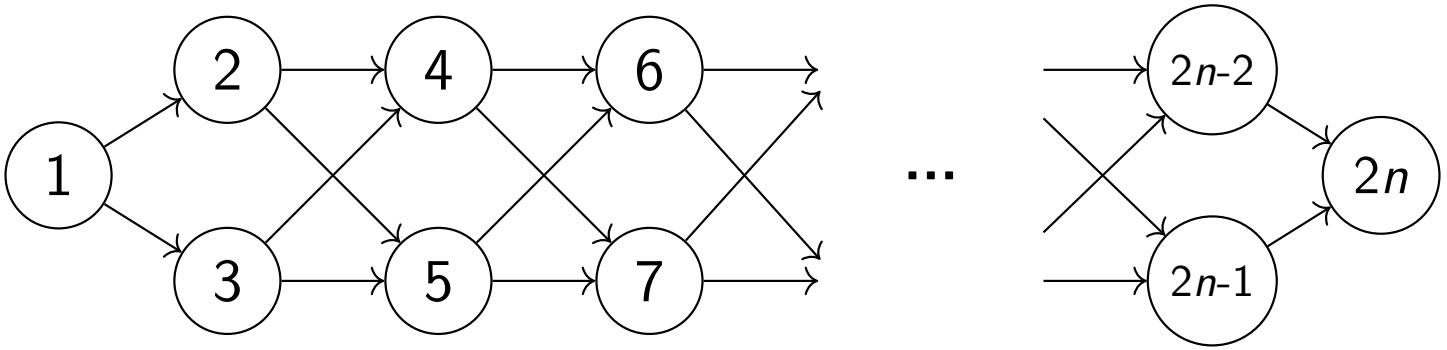
because $\begin{cases} s_1(\mathcal{L}) = ||\mathcal{L}||_2 \leq \sqrt{n}||\mathcal{L}||_\infty, \\ s_n(\mathcal{L}) = 1/||\mathcal{L}^{-1}||_2 \geq 1/\sqrt{n}||\mathcal{L}^{-1}||_\infty. \end{cases}$ □

Examples

Example 1: Random Walk fails, #Paths works.

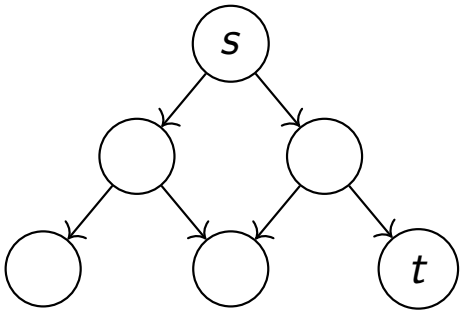


Example 2: Random Walk works, #Paths fails.

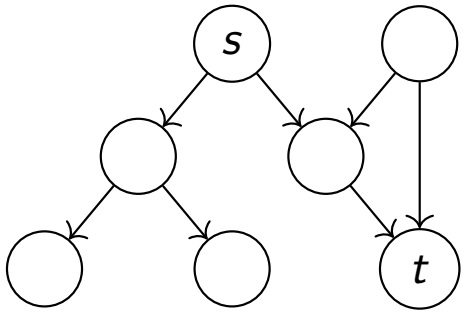


Unambiguity & Fewness

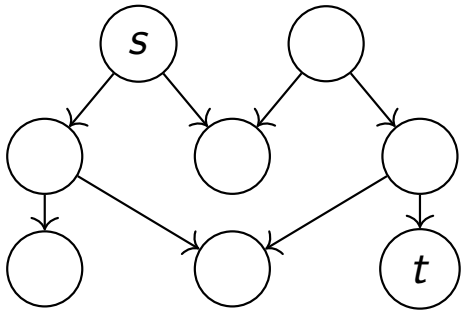
- $N(i,j) := \#\text{paths from node } i \text{ to } j$
- Following [2], a graph is called ...



unambiguous
$:\Leftrightarrow N(s,t) \leq 1$



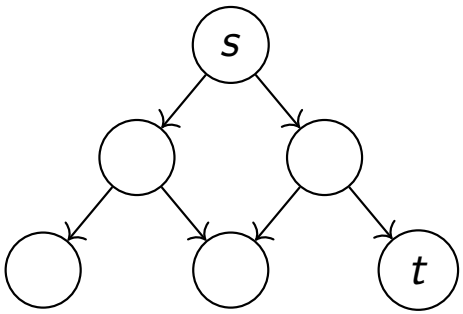
reach-unambiguous
$:\Leftrightarrow \forall j : N(s,j) \leq 1$



strongly unambiguous
$:\Leftrightarrow \forall i,j : N(i,j) \leq 1$

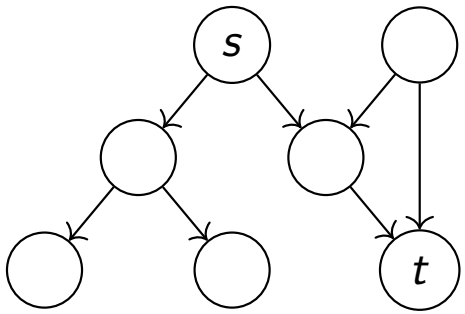
Unambiguity & Fewness

- $N(i,j) := \#\text{paths from node } i \text{ to } j$
- Following [2], a graph is called ...



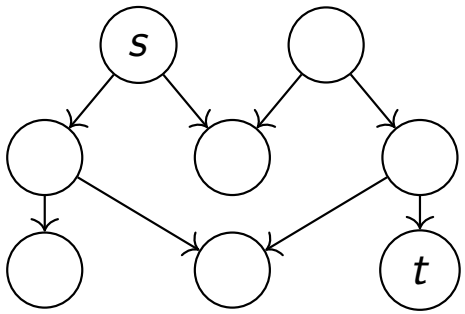
unambiguous
$:\Leftrightarrow N(s,t) \leq 1$

few unamb.
$\cdots \leq \mathrm{poly}(n)$



reach-unambiguous
$:\Leftrightarrow \forall j : N(s,j) \leq 1$

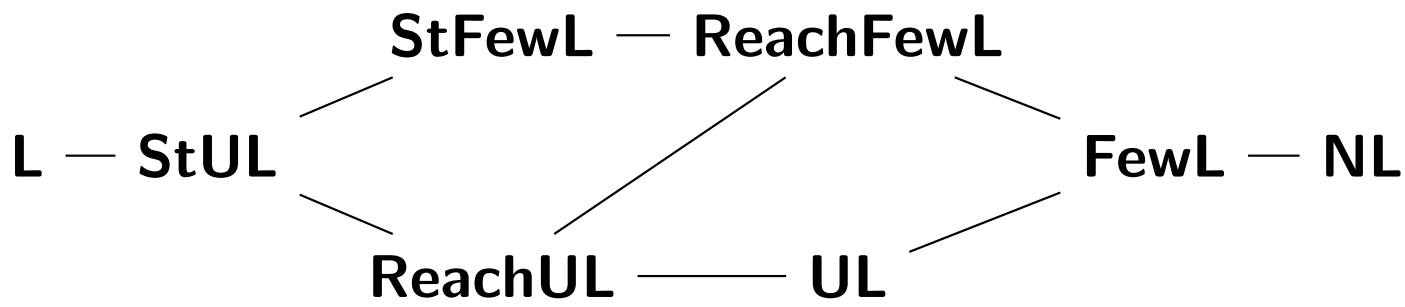
reach-few
$\cdots \leq \mathrm{poly}(n)$



strongly unambiguous
$:\Leftrightarrow \forall i,j : N(i,j) \leq 1$

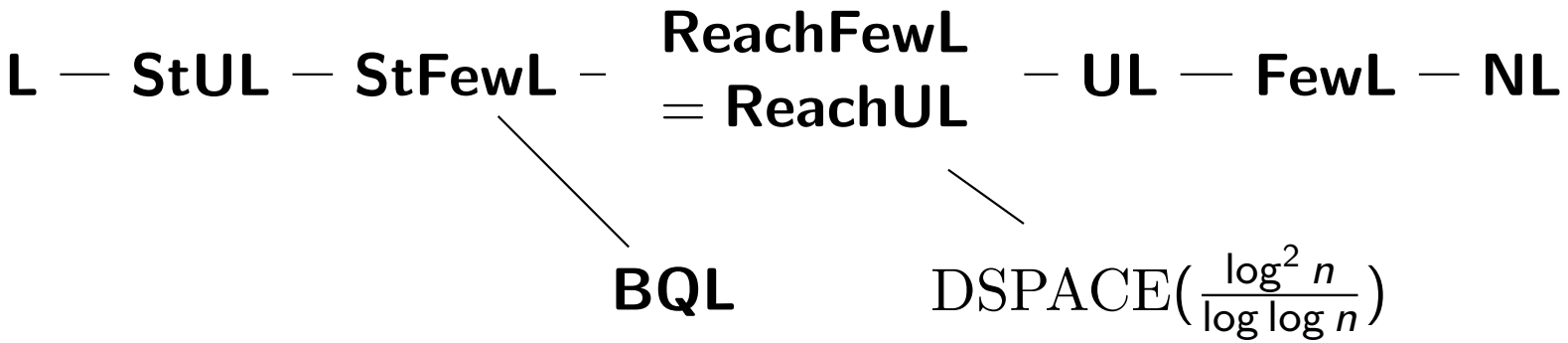
strongly-few
$\cdots \leq \mathrm{poly}(n)$

Unambiguity & Fewness



- 1998 Allender & Lange [3]: **ReachUL** $\subseteq \mathrm{DSPACE}(\frac{\log^2 n}{\log\log n})$
- 2012 Garvin, Stolee, Tewari & Vinodchandran [4]: **ReachUL** $=$ **ReachFewL**
- Previous slide: **StFewL** $\subseteq$ **BQL**

Unambiguity & Fewness



- 1998 Allender & Lange [3]: **ReachUL** $\subseteq \mathrm{DSPACE}(\frac{\log^2 n}{\log\log n})$
- 2012 Garvin, Stolee, Tewari & Vinodchandran [4]: **ReachUL** = **ReachFewL**
- Previous slide: **StFewL** $\subseteq$ **BQL**

Can we go further?

Theorem

If $\#\text{paths}(s,i)$, $\#\text{paths}(i,t) \leq \text{poly}(n)\ \forall i$, then $\text{STCON} \in \textbf{BQL}$.

Proof.

Consider again $\mathcal{L} := I - A = \sum_j s_j \left|u_j\right\rangle \left\langle v_j\right|$

$$\Rightarrow \mathcal{L}^{-1} = \underbrace{\sum_{s_j < \kappa^{-1}} s_j^{-1} \left|v_j\right\rangle \left\langle u_j\right|}_{???} + \underbrace{\sum_{s_j \geq \kappa^{-1}} s_j^{-1} \left|v_j\right\rangle \left\langle u_j\right|}_{\text{easy by Ta-Shma}}$$

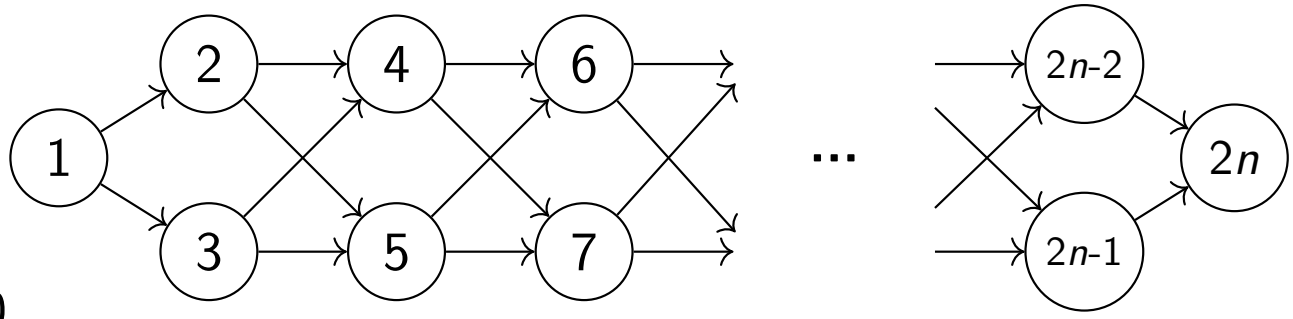
Observe $||\mathcal{L}^{-1} \left|t\right\rangle||_2^2 = \sum_j s_j^{-2} |\left\langle u_j|t\right\rangle|^2 \leq \text{poly}(n)$

$$\Rightarrow \begin{cases} |\left\langle s|v_j\right\rangle| \leq s_j \cdot \text{poly}(n), \\ |\left\langle u_j|t\right\rangle| \leq s_j \cdot \text{poly}(n) \end{cases} \Rightarrow |s_j^{-1} \left\langle s|v_j\right\rangle \left\langle u_j|t\right\rangle| \leq s_j \cdot \text{poly}(n)$$

Choose $\kappa = \text{poly}(n)$ s.t. $\sum_{s_j < \kappa^{-1}} |s_j^{-1} \left\langle s|v_j\right\rangle \left\langle u_j|t\right\rangle| \leq 1/3$. □

Open Questions

1. Can we obtain **ReachFewL** $\subseteq$ **BQL**?
2. What info can we get from ill-conditioned part?



$s_n \approx 0$

$$v_n \approx \frac{1}{*}\left[\,1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \cdots \quad \frac{1}{2^{n-1}} \quad \frac{1}{2^n}\,\right]$$

$$u_n \approx \frac{1}{*}\left[\,\frac{1}{2^n} \quad \frac{1}{2^{n-1}} \quad \frac{1}{2^{n-2}} \quad \frac{1}{2^{n-3}} \quad \cdots \quad \frac{1}{2} \quad 1\,\right]$$

Can we detect if $\frac{N(s,t)}{\sum_{i,j} N(i,j)} \geq 1/\mathrm{poly}(n)$?

3. Is there a natural **BQL**-complete graph problem? What's the relationship with #**L**, **GapL**?

$$\begin{cases} \text{pc-MATPOW}_{\mathbb{C}} \text{ is } \textbf{BQL}\text{-complete,} \\ \text{pc-MATPOW}_{\mathbb{N}} = \text{counting paths in StFew-graphs} \end{cases}$$

Conjecture: $\text{pc-MATPOW}_{\{\pm\frac{1}{2},0\}}$ is **BQL**-complete.

4. Use QSVT instead of Ta-Shma

1 Space-bounded Computation

2 Graph Connectivity

3 References

References

[1] A. Ta-Shma, "Inverting well conditioned matrices in quantum logspace," in *Proceedings of the Forty-Fifth Annual ACM Symposium on Theory of Computing*, ser. STOC '13, 2013, p. 881–890.

[2] K.-J. Lange, "An unambiguous class possessing a complete set," in *STACS 97*, R. Reischuk and M. Morvan, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 1997, pp. 339–350.

[3] E. Allender and K.-J. Lange, "RUSPACE($\log n$) $\subseteq$ DSPACE($\log^2 n / \log\log n$)," *Theory of Computing Systems*, vol. 31, 1998.

[4] B. Garvin, D. Stolee, R. Tewari, and N. V. Vinodchandran, "ReachFewL = ReachUL," in *Computing and Combinatorics*, B. Fu and D.-Z. Du, Eds., 2011, pp. 252–258.

[5] B. Fefferman and C. Y.-Y. Lin, "A complete characterization of unitary quantum space," 2016.

[6] B. Fefferman and Z. Remscrim, "Eliminating intermediate measurements in space-bounded quantum computation," in *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, ser. STOC '21, Jun. 2021.